Aging and fluctuation-dissipation ratio for the diluted Ising Model

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We consider the out-of-equilibrium, purely relaxational dynamics of a weakly diluted Ising model in the aging regime at criticality. We derive at first order in a $\sqrt{\epsilon}$ expansion the two-time response and correlation functions for vanishing momenta. The long-time limit of the critical fluctuation-dissipation ratio is computed at the same order in perturbation theory.

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According to universality hypothesis, critical phenomena can be described in terms of quantities that do not depend on the microscopic details of the systems, but only on global properties such as symmetries, dimensionality etc. A question of theoretical and experimental interest is whether and how this critical behavior is altered by introducing in the systems a small amount of uncorrelated impurities leading to models with quenched disorder.

The static critical behavior of these systems is well understood thank to the Harris criterion [1]. It states that the addition of impurities to a system which undergoes a second-order phase transition does not change the critical behavior if the specific-heat critical exponent $\alpha_{\rm p}$ of the pure system is negative. If $\alpha_{\rm p}$ is positive, the transition is altered.

For the very important class of the three-dimensional $O(M)$-vector models it is known that $\alpha_{\rm p} < 0$ for $M \geq 2$ [2], and the critical behavior is unchanged in presence of weak quenched disorder. Instead, the specific-heat exponent of the three-dimensional Ising model is positive [2], thus the existence of a new Random Ising Model (RIM) universality class is expected, as confirmed by Renormalization Group (RG) analyses, Monte Carlo simulations (MCs), and experimental works (see Refs. [2,3] for a comprehensive review on the subject, and for an updated list of references).

The purely relaxational equilibrium dynamics (Model A of Ref. [4]) of this new universality class is under intensive investigation [5–9]. The dynamic critical exponent z differs from the mean-field value already in one-loop approximation [5], at variance with the pure model. This exponent is known at three-loop level in an $\sqrt{\epsilon}$ [7] and in fixed ($d = 2, 3$) dimension [8] expansion, and has a value in good agreement with several MCs [9].

The out-of-equilibrium dynamics is instead less studied. The initial slip exponent θ of the response function was determined at two-loop order [10] and the response function only at one-loop, both for conservative and non-conservative dynamics [11].

In this work we focus on a different regime of out-of-equilibrium dynamics: the aging one. The relaxation of the system from an out-of-equilibrium initial state is characterized by two different regimes: a transient one with off-equilibrium evolution , for $t < t_R$, and a stationary one with equilibrium evolution of fluctuations for $t > t_R$, where t_R is the relaxation time. In the former a dependence of the behavior of the system on initial condition is expected, while in the latter homogeneity of time and time reversal symmetry (at least in the absence of external fields) are recovered. Consider the system in a disordered state for the initial time $t = 0$, and quench it to a given temperature $T \geq T_c$, where T_c is the critical temperature. Calling $\phi_{\mathbf{x}}(t)$ the order parameter of the model, its response to an external field h applied at a time $s > 0$ and in $\mathbf{x} = 0$ is given by the response function $R_{\mathbf{x}}(t, s) = \delta\langle\phi_{\mathbf{x}}(t)\rangle/\delta h(s)$, where $\langle\cdots\rangle$ stands for the mean over stochastic dynamics. The two-time correlation function $C_{\mathbf{x}}(t, s) = \langle\phi_{\mathbf{x}}(t)\phi_{\mathbf{0}}(s)\rangle$ is useful to describe the dynamics of order parameter fluctuations.

When the system does not reach the equilibrium (i. e. $t_R = \infty$) all the previous functions will depend both on s (the "age" of the system) and t. This behavior is usually referred to as aging and was first noted in spin glass systems [12]. To characterize the distance from equilibrium of an aging system, evolving at a fixed temperature T, the fluctuation-dissipation ratio (FDR) is usually introduced [13]:

$$X_{\mathbf{x}}(t, s) = \frac{T\,R_{\mathbf{x}}(t, s)}{\partial_s C_{\mathbf{x}}(t, s)}\,. \tag{1}$$

When $t, s \gg t_R$ the fluctuation-dissipation theorem holds and thus $X_{\mathbf{x}}(t, s) = 1$.

Recently [13–20] attention has been paid to the FDR, for nonequilibrium and nonglassy systems quenched at their critical temperature T_c from an initial disordered state [21]. The scaling form for $R_{\mathbf{x=0}}$ was rigorously established using conformal invariance [20,22]. Within the field-theoretical approach to critical dynamics, calculations are simpler if done in momentum space, thus we are interested in momentum-dependent response and correlation functions. From RG arguments it is expected that they scale, for $\mathbf{q} = \mathbf{0}$, as [23,16,18,19]

$$T_c R_{\mathbf{q=0}}(t, s) = A_R (t - s)^a (t/s)^\theta F_R(s/t)\,, \tag{2}$$

$$C_{\mathbf{q=0}}(t, s) = A_C\, s (t - s)^a (t/s)^\theta F_C(s/t)\,, \tag{3}$$

$$\partial_s C_{\mathbf{q=0}}(t, s) = A_{\partial C} (t - s)^a (t/s)^\theta F_{\partial C}(s/t)\,, \tag{4}$$

where $a = (2 - \eta - z)/z$. The functions $F_C(v)$, $F_{\partial C}(v)$ and $F_R(v)$ are universal provided one fixes the nonuniversal normalization constant A_R, A_C, and $A_{\partial C}$ to have $F_i(0) = 0$. Obviously A_C, $A_{\partial C}$, $F_C(v)$, and $F_{\partial C}(v)$ are not independent, in fact it holds

$$A_{\partial C} = A_C(1-\theta)\,, \tag{5}$$

$$F_{\partial C}(v) = F_C(v) + \frac{v}{1-\theta}\left(F_C'(v) - \frac{a}{1-v}F_C(v)\right)\,. \tag{6}$$

It has been argued that the limit $X^\infty_{\mathbf{x}=0} = \lim_{s\to\infty}\lim_{t\to\infty} X_{\mathbf{x}=0}(t,s)$ is a novel universal quantity of nonequilibrium critical dynamics [15,17] and may be computed as an amplitude ratio. Also the analog in $\mathbf{q}$ space [18]

$$\mathcal{X}_{\mathbf{q}=0}(t,s) = \frac{\Omega R_{\mathbf{q}=0}(t,s)}{\partial_s C_{\mathbf{q}=0}(t,s)} = \frac{A_R}{A_{\partial C}} = \frac{A_R}{A_C(1-\theta)} \tag{7}$$

is universal. In Ref. [18] an heuristic argument is given to show that $X^\infty_{\mathbf{x}=0} = \mathcal{X}^\infty_{\mathbf{q}=0}$.

In a recent work [19] we evaluated $X^\infty_{\mathbf{x}=0}$ at criticality for $O(M)$-vector models at the second order in the ϵ-expansion, finding values in very good agreement with two- and three- dimensional numerical simulations for the Ising model [15,17]. We also confirmed the validity of the scaling laws (2) and (4) at the same order in perturbation theory. A different model of relaxation (Model C of Ref. [4]) has also been studied [24].

The extension of this kind of investigation to disordered systems is very interesting because, besides giving a check of the expected scaling laws, it predicts a new universal dynamical quantity (the long time limit of the FDR) which could be measured in MCs and could be used to identify a universality class, as in the case of other universal quantities.

The time evolution of an N-component field $\varphi(\mathbf{x},t)$ under a purely dissipative relaxation dynamics (Model A of Ref. [4]) is described by the stochastic Langevin equation

$$\partial_t\varphi(\mathbf{x},t) = -\Omega\frac{\delta\mathcal{H}_\psi[\varphi]}{\delta\varphi(\mathbf{x},t)} + \xi(\mathbf{x},t)\,, \tag{8}$$

where Ω is the kinetic coefficient, $\xi(\mathbf{x},t)$ a zero-mean stochastic Gaussian noise with correlations

$$\langle\xi_i(\mathbf{x},t)\xi_j(\mathbf{x}',t')\rangle = 2\Omega\,\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\delta_{ij}. \tag{9}$$

and $\mathcal{H}_\psi[\varphi]$ the static Landau-Ginzburg Hamiltonian [2]

$$\mathcal{H}_\psi[\varphi] = \int \mathrm{d}^dx\left[\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}(r_0+\psi(\mathbf{x}))\varphi^2 + \frac{1}{4!}g_0\varphi^4\right]. \tag{10}$$

Here $\psi(\mathbf{x})$ is a spatially uncorrelated random field with Gaussian distribution

$$P(\psi) = \frac{1}{\sqrt{4\pi w}}\exp\left[-\frac{\psi^2}{4w}\right]. \tag{11}$$

Dynamical correlation functions, generated by Langevin equation (8) and averaged over the noise ξ, can be obtained by the field-theoretical action [25]

$$S_\psi[\varphi,\tilde\varphi] = \int \mathrm{d}t\,\mathrm{d}^dx\left[\tilde\varphi\frac{\partial\varphi}{\partial t} + \Omega\tilde\varphi\frac{\delta\mathcal{H}_\psi[\varphi]}{\delta\varphi} - \tilde\varphi\Omega\tilde\varphi\right]. \tag{12}$$

where $\tilde\varphi(\mathbf{x},t)$ is the response field.

The effect of a macroscopic initial condition $\varphi_0(\mathbf{x}) = \varphi(\mathbf{x}, t=0)$ may be taken into account by averaging over the initial configuration with a weight $e^{-H_0[\varphi_0]}$, where

$$H_0[\varphi_0] = \int \mathrm{d}^dx\,\frac{\tau_0}{2}(\varphi_0(\mathbf{x}) - a(\mathbf{x}))^2. \tag{13}$$

This specifies an initial state $a(\mathbf{x})$ with short-range correlations proportional to τ_0^{-1}.

In this way all response and correlation functions may be obtained as averages on the functional weight $\exp\{-(S_\psi[\varphi,\tilde\varphi] + H_0[\varphi_0])\}$. In the analysis of static critical behavior, the average over the quenched disorder ψ is usually performed by means of the replica trick [2]. If instead we are interested in dynamic processes it is simpler to perform directly the average at the beginning of the calculation [26]

$$\int[\mathrm{d}\psi]P(\psi)\exp(-S_\psi[\varphi,\tilde\varphi]) = \exp(-S[\varphi,\tilde\varphi]) \tag{14}$$

with the ψ-independent action

$$S[\varphi,\tilde\varphi] = \int \mathrm{d}^dx\left\{\int_0^\infty \mathrm{d}t\,\tilde\varphi\left[\partial_t\varphi + \Omega(r_0-\Delta)\varphi - \Omega\tilde\varphi\right] + \frac{\Omega g_0}{3!}\int_0^\infty \mathrm{d}t\,\tilde\varphi\varphi^3 - \frac{\Omega^2 v_0}{2}\left(\int_0^\infty \mathrm{d}t\,\tilde\varphi\varphi\right)^2\right\}, \tag{15}$$

where $v_0 \propto w$.

The perturbative expansion is performed in terms of the two fourth-order couplings g_0 and v_0 and using the propagators of the free theory with an initial condition at $t = 0$, $\langle\tilde\varphi_i(\mathbf{q},s)\varphi_j(-\mathbf{q},t)\rangle_0 = \delta_{ij}R^0_q(t,s)$ and $\langle\varphi_i(\mathbf{q},s)\varphi_j(-\mathbf{q},t)\rangle_0 = \delta_{ij}C^0_q(t,s)$ [23]. It has also been shown that τ_0^{-1} is irrelevant (in RG sense) for large times behavior [23]. From the technical point, the breaking of homogeneity in time gives rise to some problems in the renormalization procedure in terms of one-particle irreducible correlation functions (see Ref. [23] and references therein) so all the computations are done in terms of connected functions.

To compute the response function at one-loop level, we have to evaluate the two Feynman diagrams depicted in Fig. 1. In terms of them we may write

$$R_{\mathbf{q}}(t,s) = R^0(t,s) - \frac{1}{2}g_0(a) + v_0(b) + O(g_0^2, v_0^2, g_0v_0)\,, \tag{16}$$

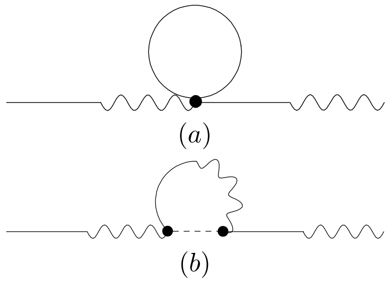


FIG. 1. Feynman diagrams contributing to the one-loop response function. Response functions are drawn as wavy-normal lines, whereas correlators are normal lines. A wavy line is attached to the response field and a normal one to the order parameter. The dotted line is a non-local v-like vertex.

where we are considering the case $N = 1$ (RIM universality class), and we set $\Omega = 1$ to lighten the notation.

In the following we report the expressions of Feynman diagrams at criticality ($r_0 = 0$ in dimensional regularization) for vanishing external momentum, since we are only interested in that limit, and since expressions for nonzero $\mathbf{q}$ are long and not very illuminating.

The diagram (a) in Fig. 1 contributes also the the response function of nondisordered models, and it has been computed in Ref. [18], obtaining (for $t > s$):

$$(a) = -N_d \frac{1}{4} \log \frac{t}{s} + O(\epsilon). \tag{17}$$

where $N_d = 2/[\Gamma(d/2)(4\pi)^{d/2}]$. For diagram (b) we find

$$\begin{aligned}(b) &= \int_0^\infty \mathrm{d}t'\,\mathrm{d}t'' \int \frac{\mathrm{d}^d p}{(2\pi)^d} R_0(t,t') R_{\mathbf{p}}(t',t'') R_0(t'',s) \\ &= \frac{1}{(4\pi)^{d/2}} \frac{1}{1-d/2} \frac{1}{2-d/2} (t-s)^{2-d/2} \,. \end{aligned}\tag{18}$$

Inserting the expression for (a) and (b) in Eq. (16) and expanding (b) at first order in ϵ, one obtains

$$\begin{aligned}R_B(t,s) = 1 + \tilde{g}_0 \frac{1}{8} \ln \frac{t}{s} - \frac{\tilde{v}_0}{2} \left[\frac{2}{\epsilon} + \log(t-s) + \gamma_E \right] \\ + O(\epsilon^2, \epsilon \tilde{g}_0, \epsilon \tilde{v}_0, \tilde{g}_0^2, \tilde{v}_0^2, \tilde{g}_0 \tilde{v}_0) \,,\end{aligned}\tag{19}$$

where $\tilde{g}_0 = N_d g_0$ and $\tilde{v}_0 = N_d v_0$. The dimensional pole in this expression can be canceled by a multiplicative renormalization of both the fields φ and $\tilde{\varphi}$.

The critical response function is then obtained setting the renormalized couplings at their fixed point values. We remind that the stable fixed point of the RIM is of order $\sqrt{\epsilon}$ and not ϵ (see, e. g. [2]), due to the degeneracy of the one-loop β functions. The nontrivial fixed point values at the first non-vanishing order (i. e. two loops) are

$$\tilde{g}_0 = \tilde{g}^* = 4 \sqrt{\frac{6\epsilon}{53}} + O(\epsilon) \,,\ \tilde{v}_0 = \tilde{v}^* = \sqrt{\frac{6\epsilon}{53}} + O(\epsilon). \tag{20}$$

Finally we get

$$R(t,s) = 1 + \frac{1}{2}\sqrt{\frac{6\epsilon}{53}} \left[\ln \frac{t}{s} - \ln(t-s) - \gamma_E \right] + O(\epsilon) \,, \tag{21}$$

that is fully compatible with the expected scaling form (2) with the well known exponents [7,11]

$$a = -\frac{1}{2}\sqrt{\frac{6\epsilon}{53}} + O(\epsilon) \quad , \quad \theta = \frac{1}{2}\sqrt{\frac{6\epsilon}{53}} + O(\epsilon) \,, \tag{22}$$

and the new results $F_R(x) = 1 + O(\epsilon)$ and

$$A_R = 1 - \frac{1}{2}\sqrt{\frac{6\epsilon}{53}} \gamma_E + O(\epsilon) \,. \tag{23}$$

There are five diagrams contributing to the correlation function. Four of them are obtained by the ones of Fig. 1 changing one of the two external response propagators with a correlation line (see Ref. [19] for a detailed explanation of this correspondence). We call these four diagrams (a_1), (a_2), (b_1), and (b_2). The sum $(a_1) + (a_2)$ was computed in [18] leading to

$$(a_1) + (a_2) = -\frac{N_d}{2} s \left(\log \frac{t}{s} + 2 \right) + O(\epsilon) \,. \tag{24}$$

The sum $(b_1) + (b_2)$ is instead

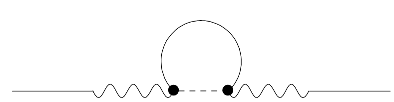
$$\begin{aligned}(b_1) + (b_2) = \frac{N_d \Gamma(d/2)}{(1-d/2)(2-d/2)(3-d/2)} \times \\ [t^{3-d/2} + s^{3-d/2} - (t-s)^{3-d/2}].\end{aligned}\tag{25}$$

The octopus diagram in Fig 2 does not have a corresponding one contributing to the response function. It has the value

$$\begin{aligned}(c) = \frac{N_d \Gamma(d/2)}{(1-d/2)(2-d/2)(3-d/2)} \times \\ \left[\frac{(t-s)^{3-d/2} + (t+s)^{3-d/2}}{2} - t^{3-d/2} - s^{3-d/2} \right] .\end{aligned}\tag{26}$$

Collecting together these contributions and expanding in powers of ϵ we find, at $O(\epsilon^2, \epsilon \tilde{g}_0, \epsilon \tilde{v}_0, \tilde{g}_0^2, \tilde{v}_0^2, \tilde{g}_0 \tilde{v}_0)$,

$$\begin{aligned}C_B(t,s) &= 2s - \frac{g_0}{2}((a_1) + (a_2)) + v_0((b_1) + (b_2) + (c)) \\ &= 2s + \frac{\tilde{g}_0}{4} s \left(\log \frac{t}{s} + 2 \right) + \tilde{v}_0 \left[-\frac{2s}{\epsilon} - \gamma_E s \right. \\ &\left. + s + \frac{(t-s)\log(t-s)}{2} - \frac{(t+s)\log(t+s)}{2} \right] .\end{aligned}\tag{27}$$



(c)

FIG. 2. Feynman diagram contributing to the one-loop correlation function that does not have analog in the response.

Also this expression is renormalized by canceling the dimensional pole by means of multiplicative renormalizations of parameters and fields. At the nontrivial fixed point for couplings we have

$$\frac{C(t,s)}{2} = s\left\{1+\frac{1}{2}\sqrt{\frac{6\epsilon}{53}}\left[\log\frac{t}{s}+3-\gamma_E-\log(t-s)\right.\right.$$
$$\left.\left.+\frac{t+s}{2s}\log\frac{t-s}{t+s}\right]\right\}+O(\epsilon)\ , \tag{28}$$

which is again compatible with the expected scaling form (cf. Eq. (3)) with the same exponents of Eq. (22), the non universal amplitude

$$\frac{A_C}{2} = 1+\frac{1}{2}\sqrt{\frac{6\epsilon}{53}}(2-\gamma_E)+O(\epsilon)\,, \tag{29}$$

and the universal regular scaling function

$$F_C(x) = 1+\frac{1}{2}\sqrt{\frac{6\epsilon}{53}}\left[1+\frac{1}{2}\left(1+\frac{1}{x}\right)\log\frac{1-x}{1+x}\right]+O(\epsilon)\,. \tag{30}$$

Note that at variance with the pure model [18,19], the function $F_C(x)$ receives a contribution already at one-loop order which should be observable in MCs.

Using Eq. (7) the FDR is

$$\mathcal{X}^{\infty}_{\mathbf{q}=0} = \frac{1}{2}-\frac{1}{4}\sqrt{\frac{6\epsilon}{53}}+O(\epsilon)\ , \tag{31}$$

that, for $\epsilon=1$ leads to $\mathcal{X}^{\infty}_{\mathbf{q}=0}\sim 0.416$, and ~ 0.381 for $\epsilon=2$. It would be interesting to see if this one-loop result is in as good agreement with MCs as in the case of the pure model (cf. Refs. [18,19]). To this order it is not even clear whether randomness really changes in a sensible way the limit of the FDR or not. Two-loop computations and MCs could clarify this point.

For completeness we report also the FDR for finite times:

$$\frac{\mathcal{X}^{-1}_{\mathbf{q=0}}(t,s)}{2} = 1+\frac{1}{2}\sqrt{\frac{6\epsilon}{53}}\left[1+\frac{1}{2}\log\frac{t-s}{t+s}\right]+O(\epsilon)\ . \tag{32}$$

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